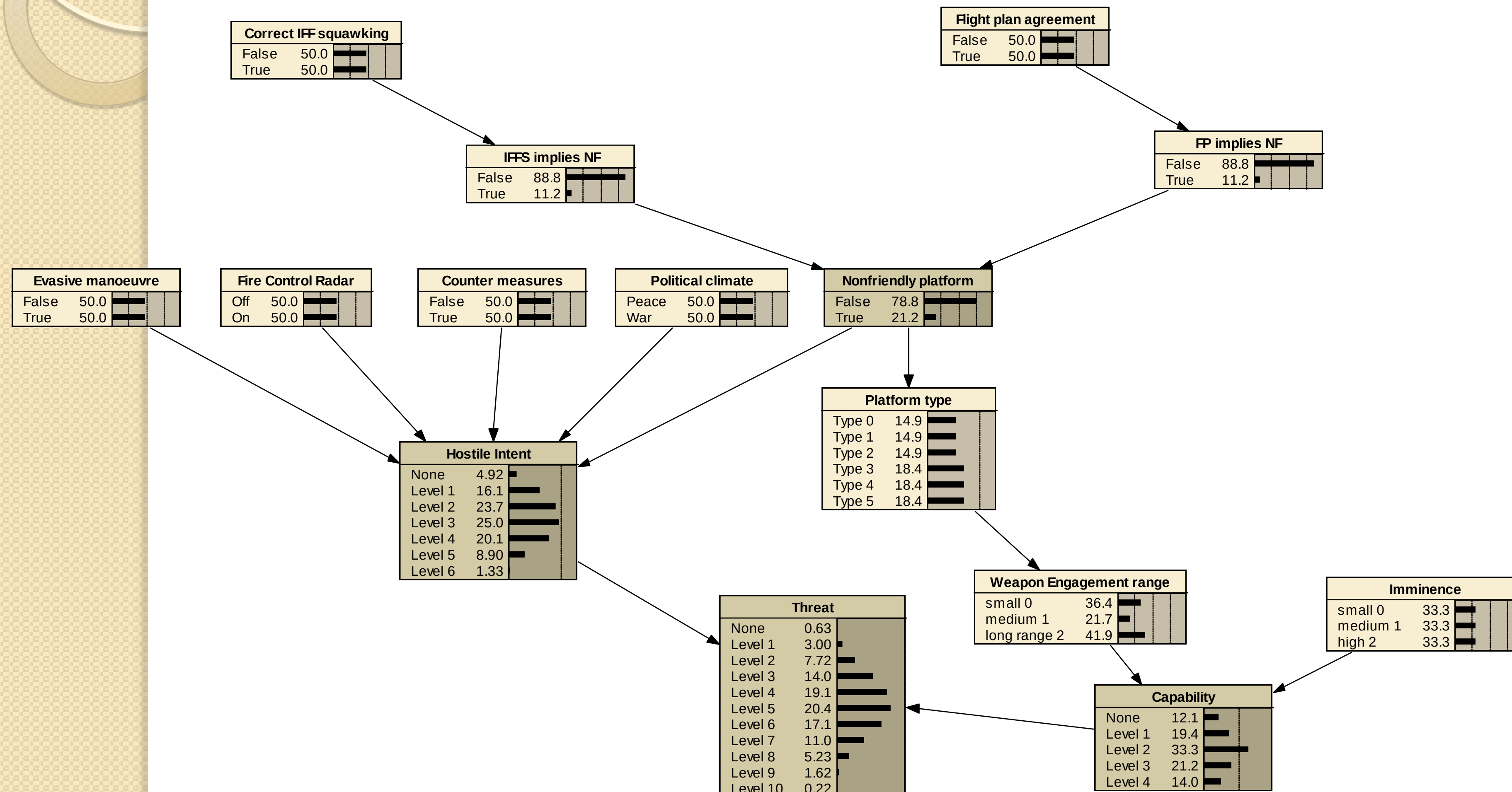
# An Experimental Design Approach to Sensitivity Analysis

David Albrecht
Monash University

# Overview

- Threat Assessment Model
- Analysis of the model
- Conclusions

# Threat Assessment Model




A. Benavoli, B. Ristic, A. Farina, M. Oxenham, L. Chisci, *An Approach to Threat Assessment Based on Evidential Networks,* Information Fusion, 2007

# Standard Analysis

- Uncertainty metrics:
  - Mutual Information

$$\sum_{t}\sum_{q} \mathrm{Pr}(\mathrm{T}=\mathrm{t},\mathrm{Q}=\mathrm{q}) \log\left(\frac{\mathrm{Pr}(T=t,Q=q)}{\mathrm{Pr}(T=t)\,\mathrm{Pr}(Q=q)}\right)$$

  - Variance of beliefs

$$\sum_{t}\sum_{q} \mathrm{Pr}(T=t,Q=q)(\mathrm{Pr}(T=t|Q=q)-\mathrm{Pr}(T=t))^2$$

- No interactions

# Standard Sensitivity Analysis

| Variable | Mutual Information | Percentage of Entropy | Variance of Beliefs |
|---|---|---|---|
| **T** | 2.92831 | 100.00 | 0.727401 |
| **HI** | 0.70133 | 23.90 | 0.013128 |
| **C** | 0.44000 | 15.00 | 0.005787 |
| **FCR** | 0.25023 | 8.55 | 0.003402 |
| **WER** | 0.20874 | 7.13 | 0.002513 |
| **I** | 0.16313 | 5.57 | 0.001820 |
| **PT** | 0.05875 | 2.01 | 0.000759 |
| **PC** | 0.05782 | 1.97 | 0.000585 |
| **EM** | 0.05782 | 1.97 | 0.000585 |
| **CM** | 0.05782 | 1.97 | 0.000585 |
| **NF** | 0.05348 | 1.83 | 0.000563 |
| **IFFS** | 0.00197 | 0.0674 | 0.000020 |
| **FPA** | 0.00197 | 0.0674 | 0.000020 |

# Issues

- Not interested in measures of uncertainty
- Would like to know the sensitivity of the Expected Level of Threat, E(T).
- Need to know whether interactions are important.

# New Approach

- Choose the statistic.
- Choose an Experimental Design.
- Collect data.
- Build an empirical model.
- Determine the main effects and interactions of the model.
- Decompose the variance of the statistic.

# Expected Threat Level

- Want to model E(T) in terms of:
  - Evasive manoeuvre (EM)
  - Fire Control Radar (FCR)
  - Countermeasures (CM)
  - Political Climate (PC)
  - Imminence (I)
  - Correct IFF Squawking (IFFS)
  - Flight Plan Agreement (FPA).

# Choose Experimental Design

Some options are:

- Factorial Design
  - All possible combinations of values of the variables.
- Fractional Factorial Design
  - Subset of possible combinations of values of the variables.
- Latin Hypercube
  - Combinations of values of the parameters.

# Collect Data

For the values in the experimental design

- Set the values in the Bayesian Network
- Update your belief of T
- Calculate the statistic, i.e., E(T).

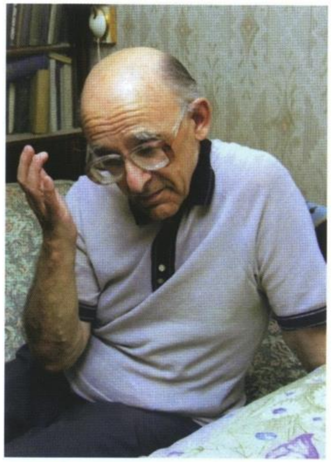
# Example

- Choose a Factorial Design
- Collected 192 values of E(T)
- Choose a two factor interaction model

$$y = \beta_0 + \sum_{j=1}^{n} \beta_j x_j + \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} \beta_{j,k} x_j x_k + \varepsilon$$

- Used least squares to fit the model

# Sobol Decomposition



$$y = f(x_1, x_2, \cdots, x_n)$$

where $0 \le x_1 \le 1, \cdots, 0 \le x_n \le 1$

Let

$$y_0 = \int_0^1 \cdots \int_0^1 f(x_1, x_2, \cdots, x_n)\, dx_1 \cdots dx_n$$

Main Effects

$$y_k(x_k) = \int_0^1 \cdots \int_0^1 f(x_1, x_2, \cdots, x_n) d\, x_{-k} - y_0$$

Two factor interactions

$$y_{j,k}(x_j, x_k) = \int_0^1 \cdots \int_0^1 f(x_1, x_2, \cdots, x_n)\, d\, x_{-jk} - y_j(x_j) - y_k(x_k) - y_0$$

Multidimensional Integrations

Then

$$y = y_0 + \sum_{k=1}^{n} y_k(x_k) + \sum_{k=1}^{n-1} \sum_{j=k+1}^{n} y_{j,k}(x_j, x_k) + \cdots$$

# Second order interaction model Sobol decomposition

$$y = \beta_0 + \sum_{j=1}^{n} \beta_j x_j + \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} \beta_{j,k} x_j x_k + \varepsilon$$

$$y_0 = \beta_0 + \frac{1}{2} \sum_{k=1}^{n} \beta_k + \frac{1}{4} \sum_{k=1}^{n-1} \sum_{j=k+1}^{n} \beta_{j,k}$$

$$y_k(x_k) = \alpha_k (x_k - \frac{1}{2})$$

where $\alpha_k = \beta_k + \frac{1}{2} \sum_{i=1}^{k-1} \beta_{k,i} + \frac{1}{2} \sum_{i=k+1}^{n} \beta_{i,k}$

$$y_{j,k}(x_j, x_k) = \beta_{j,k} (x_j - \frac{1}{2})(x_k - \frac{1}{2})$$

# Decomposition of E(T)

$$E(T) \cong 1.443 + EM + 2 * FCR + \ CM + PC + 2I' - 0.166 * FPA$$
$$- 0.166 * IFFS \ - 0.0362 * FPA * IFFS$$

where I = 2*I'.

So

,

| | |
|---|---|
| $y_{EM}$ | = EM – 0.5 |
| $y_{FCR}$ | = 2(FCR-0.5) |
| $y_{CM}$ | = CM-0.5 |
| $y_{PC}$ | = PC – 0.5 |
| $y_{I'}$ | = 2(I' – 0.5) |
| $y_{FPA}$ | = -0.2022(FPA – 0.5) |
| $y_{IFFS}$ | = -0.2022(IFFS - 0.5) |
| $y_{IFFS,FPA}$ | = 0.0362(FPA-0.5)(IFFS-0.5) |

Interaction

$$E(T) = 1.443 + \ y_{EM} + y_{FCR} + y_{CM} + y_{PC} + y_{I'} + y_{FPA} + y_{IFFS} + y_{IFFS,FPA}$$

Main Effects

# Variance Decomposition

$$y = y_0 + \sum_{k=1}^{n} y_k\,(x_k) + \sum_{k=1}^{n-1} \sum_{j=k+1}^{n} y_{j,k}(x_j, x_k) + \cdots$$

$$Var(y) = \sum_{k=1}^{n} Var(y_k) + \sum_{k=1}^{n-1} \sum_{j=k+1}^{n} Var(y_{j,k}) + \cdots$$

where

$$Var(y) = \int_0^1 \cdots \int_0^1 (y - y_0)^2 \; dx_1 \cdots dx_n$$

$$Var(y_k) = \int_0^1 y_k(x_k)^2 dx_k$$

$$Var(y_{j,k}) = \int_0^1 \int_0^1 y_{j,k}(x_j, x_k)^2 dx_j dx_k$$

# Variance Decomposition of E(T)

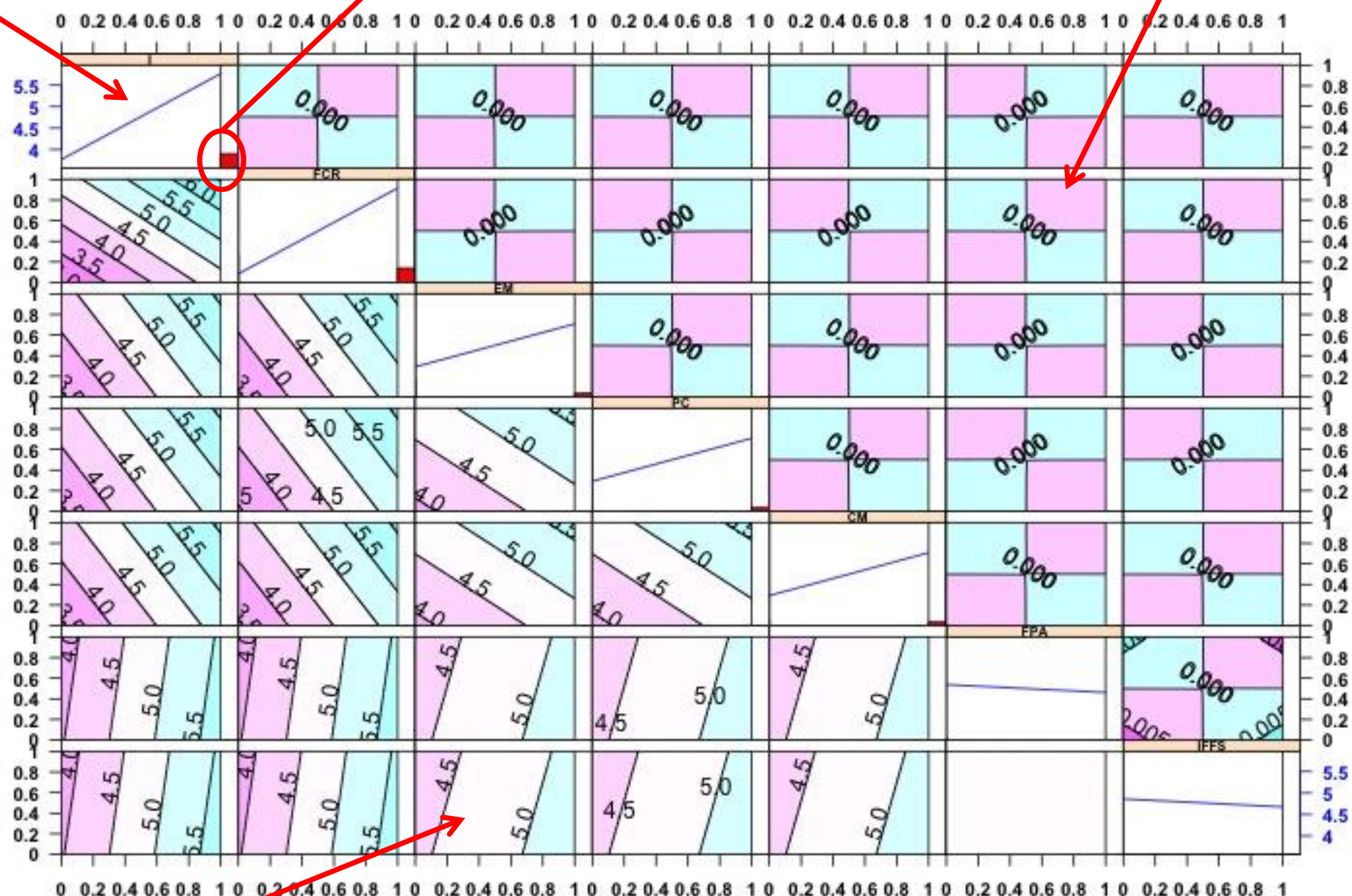
| Effect | Variance |
|---|---|
| $y_{EM}$ = EM – 0.5 | 0.08333 |
| $y_{FCR}$ = 2(FCR-0.5) | 0.33333 |
| $y_{CM}$ = CM-0.5 | 0.08333 |
| $y_{PC}$ = PC – 0.5 | 0.08333 |
| $y_{I'}$ = 2(I' – 0.5) | 0.33333 |
| $y_{FPA}$ = -0.2022(FPA – 0.5) | 0.00340 |
| $y_{IFFS}$ = -0.2022(IFFS - 0.5) | 0.00340 |
| $y_{IFFS,FPA}$ = 0.0362(FPA-0.5)(IFFS-0.5) | 0.00001 |

# Visualization of Sensitivity

Main effect

Total variance of effect

Interaction effects




Joint effects

# Conclusions

- Experimental Designs and Bayesian updating can be used to find empirical models.
- Sobol decomposition can be used to perform sensitivity analysis.
- Visualization is useful for understanding the variance, main effects, interactions and joint effects.